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Discrete-time Convolution Feedback Systems†

by

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Abstract

This paper considers multi-input multi-output discrete-time feedback systems characterized by $y = G*e$ and $e = u-y$. Theorem I shows that if the closed loop impulse response H is stable in the sense that $H \in \ell^1_{n\times n}(\rho)$, then $\tilde{G}(z) = \tilde{N}(z)[\tilde{D}(z)]^{-1}$ where $\tilde{N}(z)$, $\tilde{D}(z)$ are in $\tilde{\ell}^1_{n\times n}(\rho)$. Theorem II gives <u>necessary and sufficient</u> conditions for $H \in \ell^1_{n\times n}(\rho)$. Finally Theorem III gives <u>necessary and sufficient</u> conditions for stability when $\tilde{G}(z)$ has a finite number of multiple poles in $|z| \geq \rho$: the case where the leading term of the Laurent expansion at each of these poles is singular is treated in detail.

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I Introduction

This paper considers discrete-time linear time-invariant feedback systems with n inputs and n outputs. It is of course closely related to the corresponding paper [1] which deals with the continuous-time case. In view of the simpler analytic nature of the present problems, some of the results are sharper and the proofs use more elementary tools. For the convenience of the reader, the present paper is as self-contained as possible.

For the feedback system under consideration, the input u, output y and error e are sequences mapping $\mathbb{Z}_+$ (the set of nonnegative integers) into $\mathbb{R}^n$. The open loop system is of the convolution type so that we have

(1) $$y = G*e$$

(2) $$e = u-y \ .$$

G is specified by a sequence of real n×n matrices $\{G_i\}_{i=0}^{\infty}$; thus (1) is equivalent to $y_m = \sum_{i=0}^{\infty} G_{m-i}\, e_i$, for m = 0,1,2,... . We use $\underset{\sim}{G}$ to denote the map $\underset{\sim}{G}$: $e \mapsto G*e$. As it will become apparent, there is no loss of generality in taking the feedback to be unity as we did in (2).

We shall repeatedly use the convolution algebra $\ell^1(\rho)$; f is said to be in $\ell^1(\rho)$ iff

(3) $$f = (f_0, f_1, f_2, \ldots)$$

where $f_i \in \mathbb{R}$ for all i and $\sum_{i=0}^{\infty} |f_i| \rho^{-i} < \infty$. The product of two elements $f, g \in \ell^1(\rho)$ is given by their convolution: $(f*g)_m = \sum_{i=0}^{m} f_{m-i}\, g_i$ and it is easy to show $f*g \in \ell^1(\rho)$. The case $\rho = 1$, is handled in [2]. An n vector v (n×n matrix A) is said to be in $\ell^1_n(\rho)$ ($\ell^1_{n\times n}(\rho)$) iff all its elements are in $\ell^1(\rho)$. Let $\tilde{f}$ denote the z-transform of f, i.e. $\tilde{f}(z) = \sum_{i=0}^{\infty} f_i\, z^{-i}$: f belongs to the convolution algebra $\ell^1(\rho)$ iff $\tilde{f}$ belongs to the algebra $\tilde{\ell}^1(\rho)$ (with pointwise product). Similarly $\tilde{v} \in \tilde{\ell}^1_n(\rho)$, $\tilde{A} \in \tilde{\ell}^1_{n\times n}(\rho)$.

One of the most interesting results of this paper is to show the overwhelming importance of systems described by (1) and (2) where

$$\tilde{G}(z) = \tilde{N}(z)[\tilde{D}(z)]^{-1} \tag{4}$$

with $\tilde{N}, \tilde{D} \in \tilde{\ell}^1_{n\times n}(\rho)$. This class has been studied by M. Vidyasagar [4]. In theorem I below it is proved that once the closed loop impulse response H is well defined, then, if $H \in \ell^1_{n\times n}(\rho)$, it follows that $\tilde{G}$ is of the form (4). Theorem I uses an observation of Nasburg and Baker [5] who considered single-input single-output continuous-time systems. Theorem II is a straightforward extension of a result of [5]: it shows the importance of the systems considered by Vidyasagar in the sense that $H \in \ell^1_{n\times n}(\rho)$ if and only if $\tilde{G}$ is of the form (4). Finally theorem III gives the necessary and sufficient conditions for stability of the closed loop system when $\tilde{G}$ is of the form (4) with a finite number of poles in $|z| \geq \rho$. This work completely solves the problem considered in the recent papers of Desoer, Wu and Lam [2,3].

II. The Relation Between $\underset{\sim}{G}$ and $\underset{\sim}{H}$.

We shall use repeatedly following lemma

Lemma I

Let $\tilde{A} \in \tilde{\ell}^1_{n\times n}(\rho)$, then $\tilde{A}^{-1} \in \tilde{\ell}^1_{n\times n}(\rho)$ if and only if $\inf_{|z|\geq\rho} |\det \tilde{A}(z)| > 0$.

This is easy to establish by slightly modifying the proof of lemma 2 in [2].

Theorem I

Let G be a sequence of real n×n matrices $\{G_i\}_{i=0}^{\infty}$. For the system defined by (1) and (2) assume that the closed-loop impulse response exists and is uniquely defined by

(5) $$H + G*H = G .$$

Under these conditions, if $H \in \ell^1_{n\times n}(\rho)$, then

(a) G is z-transformable and for some finite $\bar{\rho} \geq \rho$, $G \in \ell^1_{n\times n}(\bar{\rho})$.

(b) $\tilde{G}$ is of the form

(6) $$\tilde{G}(z) = \tilde{N}(z)[\tilde{D}(z)]^{-1} \text{ for } |z| > \rho$$

where $\tilde{N}(\cdot)$ and $\tilde{D}(\cdot) \in \tilde{\ell}^1_{n\times n}(\rho)$.

(c) $\tilde{G}$ can at most have a finite number of poles in any annulus of the form $\rho + \varepsilon \leq |z| \leq \bar{\rho}$ where $\varepsilon > 0$ with $\rho + \varepsilon < \bar{\rho}$.

Comment

This theorem shows that once the closed loop system is well-defined and "stable", then $\tilde{G}$ is necessarily of the form (6), can at most have a finite number of poles in any annulus of the form $\rho + \varepsilon \leq |z| \leq \bar{\rho}$ and is analytic for $|z| > \bar{\rho}$.

Proof

Note that H is of the form $H = (H_0, H_1, \ldots)$ and $\sum_{i=0}^{\infty} |H_i| \rho^{-i} < \infty$, where $|\cdot|$ denotes any matrix norm. From (5), $H_0 + G_0 H_0 = G_0$ so that, since H_0 is uniquely defined, $\det(I+G_0) \neq 0$. This implies $\det(I-H_0) \neq 0$ because (5) implies that $(I+G_0)(I-H_0) = I$. Since $\tilde{H}(z) \to H_0$ as $|z| \to \infty$, there exists a finite $\bar{\rho} \geq \rho$ such that $\det[I-\tilde{H}(z)] \neq 0$ for $|z| > \bar{\rho}$. Since $\tilde{G}(z) = \tilde{H}(z)[I-\tilde{H}(z)]^{-1}$, conclusion (a) follows and, using analytic continuation into the annulus $\rho < |z| \leq \bar{\rho}$, (6) follows with $\tilde{N} = \tilde{H}$ and $\tilde{D} = I-\tilde{H}$. Since by assumption $H \in \ell^1_{n\times n}(\rho)$, $\tilde{N}$ and $\tilde{D} \in \ell^1_{n\times n}(\rho)$, so conclusion (c) follows easily by contradiction; note that the elements of $\tilde{D}$ are analytic in the compact annulus $\rho + \varepsilon \leq |z| \leq \bar{\rho}$. ¤

Remark

1) Observe that under the conditions of theorem I we have $[I+\tilde{G}(z)][I-\tilde{H}(z)] = I$ for $|z| > \rho$. Thus $\underset{\sim}{H}$ and $\underset{\sim}{G}$ play a symmetrical role:
$\underset{\sim}{H}$ is obtained from $\underset{\sim}{G}$ by negative feedback of I;
$\underset{\sim}{G}$ is obtained from $\underset{\sim}{H}$ by negative feedback of -I.

2) A little more can be said about the poles of $\tilde{G}(z)$:

$$G(z) = \tilde{N}(z)[\tilde{D}(z)]^{-1} = \tilde{N}(z)\mathrm{Adj}[\tilde{D}(z)]/\det[\tilde{D}(z)] .$$

The function ϕ: $z \mapsto \det[\tilde{D}(z)]$ is analytic and bounded in $|z| > \rho$; furthermore, $\lim_{|z|\to\infty} \det[\tilde{D}(z)] = \det D_0 = \det(I-H_0) \neq 0$. Therefore, by a theorem of [9], ϕ has either a finite number of zeros in $|z| > \rho$, or else it has an infinite sequence of them $\{p_i\}_{i=1}^{\infty}$ such that $\sum_{i=1}^{\infty}(1-\rho/|p_i|) < \infty$. Thus $\tilde{G}(z)$ has either a finite number of poles in $|z| > \rho$ or else it has an infinite sequence of them and all accumulation points of this sequence lie on $|z| = \rho$.

Theorem II

Let G be a sequence of n×n real matrices which is z-transformable. For the system defined by (1) and (2) assume that the closed-loop transfer function $\tilde{H}$ is well-defined for almost all z in the domain of convergence of $\tilde{G}$; more precisely,

(7) $$\tilde{H}(z) = \tilde{G}(z)[I+\tilde{G}(z)]^{-1}$$

for almost all z in the domain of convergence of $\tilde{G}(\cdot)$. Under these conditions,

(8) $$H \in \ell^1_{n\times n}(\rho)$$

if and only if there exists $\tilde{N}, \tilde{D} \in \tilde{\ell}^1_{n\times n}(\rho)$ such that

(9) $$\tilde{G}(z) = \tilde{N}(z)[\tilde{D}(z)]^{-1}$$

and

(10) $$\inf_{|z|\geq\rho} |\det[\tilde{N}(z)+\tilde{D}(z)]| > 0 .$$

Proof

Necessity By (8), $\tilde{H}(z)$ is analytic and bounded in $|z| > \rho$. Equation (7) is equivalent to $\tilde{G}(z) = \tilde{H}(z)[I-\tilde{H}(z)]^{-1}$, so (9) and (10) follow with $\tilde{N} = \tilde{H}$, $\tilde{D} = I-\tilde{H}$.

Sufficiency By assumption $\tilde{N}$ and $\tilde{D} \in \tilde{\ell}^1_{n\times n}(\rho)$ and (7) gives $\tilde{H}(z) = \tilde{N}(z)[\tilde{N}(z)+\tilde{D}(z)]^{-1}$. By lemma I, (10) implies that the second factor is in $\tilde{\ell}^1_{n\times n}(\rho)$. Therefore $\tilde{H}(z) \in \tilde{\ell}^1_{n\times n}(\rho)$ as a product of two factors in this algebra. ¤

Remark

As in the continuous-time case, (9) does not determine the ordered pair $(\tilde{N},\tilde{D})$ uniquely for a given $\tilde{G}$. In order that condition (10) depend only on $\tilde{G}$ we may, as Vidyasagar, impose on the pair $(\tilde{N},\tilde{D})$ a no-cancellation condition [1,4].

III. Necessary and Sufficient Conditions for Stability.

We consider first and in detail the case where $\tilde{G}$ has a single pole p of order m in $|z| \geq \rho$.

We consider the open loop transfer function

(11) $$\tilde{G}(z) = \sum_{i=0}^{m-1} R_i \, (z-p)^{-m+i} + G_0(z)$$

where $|p| \geq \rho > 0$, $\tilde{G}_0(z) \in \tilde{\ell}^1_{n\times n}(\rho)$, $r_0 = \text{rank } R_0 \leq n$ and R_i are $n\times n$ matrices with complex coefficients. To streamline proofs, we state some preliminary facts.

Fact 1. Let

$$(12) \qquad \tilde{\hat{R}}(1/z) \triangleq \left(\sum_{i=0}^{m-1} R_i(z-p)^{-m+i}\right)\left((z-p)/z\right)^m$$

then $\tilde{\hat{R}}(1/z)$ is a polynomial matrix in $(1/z)$ of degree m.

Fact 2. (Smith Canonical form [7]). For the n×n polynomial matrix $\tilde{\hat{R}}(1/z)$ there exist unimodular (i.e. with nonzero constant determinant) polynomial matrices in $(1/z)$ viz. $\tilde{P}(1/z)$ and $\tilde{Q}(1/z)$, such that:

$$(13) \qquad \tilde{Q}(1/z)\tilde{\hat{R}}(1/z)\tilde{P}(1/z) =$$

$$\mathrm{diag}\{\underbrace{\tilde{a}_1(1/z),\ldots,\tilde{a}_j(1/z),\ldots,\tilde{a}_r(1/z)}_{r},\ \underbrace{0,0,\ldots,0}_{n-r}\}$$

where i) r = rank of $\tilde{\hat{R}}(1/z)$ = order of the largest minor of $\tilde{\hat{R}}(1/z)$ whose determinant is not equal to the zero polynomial; ii) $\tilde{a}_j(1/z)$, $j = 1,2,\ldots,r$ are the invariant polynomials of $\tilde{\hat{R}}(1/z)$ and each polynomial $\tilde{a}_j(\cdot)$ divides $\tilde{a}_{j+1}(\cdot)$, $j = 1,2,\ldots, r-1$; iii) the diagonal matrix on the R.H.S. of (13) can be obtained by elementary operations.

Fact 3. The polynomial matrices $\tilde{P}(1/z)$ and $\tilde{Q}(1/z) \in \tilde{\ell}^1_{n\times n}(\rho)$ and their inverses are polynomial matrices also in $\tilde{\ell}^1_{n\times n}(\rho)$.

Fact 4. Let $\tilde{a}_j(\cdot)$ $j = 1,2,\ldots,r$ be as in (13) and let r_0 be the rank of R_0, then

(a)

$$(14)\qquad \begin{cases} \tilde{a}_j(1/p) \neq 0 \text{ for } 1 \leq j \leq r_0; \\ \tilde{a}_j(1/p) = 0 \text{ for } r_0 + 1 \leq j \leq r; \end{cases}$$

(b) by the factorization of the last $r-r_0$ polynomials

$$(15)\qquad \tilde{a}_j(1/z) = \tilde{b}_j(1/z)((z-p)/z)^{c_j} \text{ for } r_0 + 1 \leq j \leq r,$$

where c_j is the order of the zero of $\tilde{a}_j(\cdot)$ at $z = p$; $\tilde{b}_j(\cdot)$ is a polynomial with

$$(16)\qquad \tilde{b}_j(1/p) \neq 0, \text{ and}$$

$$1 \leq c_{r_0+1} \leq c_{r_0+2} \leq \cdots \leq c_r.$$

<u>Remark</u>

The c_j may be larger than m (in fact $c_r \leq rm$) and are monotonically increasing. Thus the c_j-m's may take on any sign. Therefore partition the index set $K = \{r_0+1, r_0+2,\ldots,r\}$ into:

$$(17)\qquad K_- = \{r_0+1, r_0+2,\ldots,\alpha\} = \{j \mid 1 \leq c_j < m\}$$

$$(18)\qquad K_0 = \{\alpha+1, \alpha+2,\ldots,\beta\} = \{j \mid c_j = m\}$$

$$(19)\qquad K_+ = \{\beta+1, \beta+2,\ldots,r\} = \{j \mid c_j > m\} \quad .$$

We are now ready for theorem III.

Theorem III

Let $\tilde{G}(z)$ be given by (11) and let $\tilde{P}(1/z)$ and $\tilde{Q}(1/z)$ be the polynomial matrices defined in (13). Suppose that the index-sets K_-, K_0, K_+, as defined in (17)-(19) are not empty.

Consider the partitioning

(20) $$\tilde{Q}(1/z)[I+\tilde{G}_0(z)]\tilde{P}(1/z) = \begin{array}{c c} & \begin{array}{cc} \overbrace{\phantom{\tilde{L}_{11}(z)}}^{\alpha} & \overbrace{\phantom{\tilde{L}_{12}(z)}}^{n-\alpha} \end{array} \\ \begin{array}{c} \alpha \, \{ \\ n-\alpha \, \{ \end{array} & \left[\begin{array}{c|c} \tilde{L}_{11}(z) & \tilde{L}_{12}(z) \\ \hline \tilde{L}_{21}(z) & \tilde{L}_{22}(z) \end{array}\right] \end{array}$$

and let $\tilde{b}_j(\cdot)$ be the polynomials defined in (15). Under these conditions,

(21) $$H \in \ell^1_{n\times n}(\rho)$$

if and only if

(22) $$\inf_{|z|\geq\rho} |\det[I+\tilde{G}(z)]| > 0$$

and at the pole p the following condition holds

(C) $$\det\{\tilde{L}_{22}(p) + \mathrm{diag}[\tilde{b}_{\alpha+1}(1/p),\ldots,\tilde{b}_{\beta}(1/p),0,0,\ldots,0]\} \neq 0.$$

Proof

Sufficiency. Note that (21) is equivalent to $[I+\tilde{G}(z)]^{-1} \in \tilde{\ell}^1_{n\times n}(\rho)$, which by fact 3 is again equivalent to $\{\tilde{Q}(1/z)[I+\tilde{G}(z)]\tilde{P}(1/z\}^{-1} \in \tilde{\ell}^1_{n\times n}(\rho)$. Take now as multiplier

$\tilde{M}(z) =$

$$\text{(23)}\qquad \mathrm{diag}\{\underbrace{\tilde{s}(z)^{m},\tilde{s}(z)^{m},\ldots,\tilde{s}(z)^{m}}_{r_0},\ \underbrace{\tilde{s}(z)^{m-c_{r_0+1}},\tilde{s}(z)^{m-c_{r_0+2}},\ldots,\tilde{s}(z)^{m-c_\alpha}}_{\alpha-r_0},$$

$$\underbrace{1,1,\ldots,1}_{n-\alpha}\}$$

with

$$\text{(24)}\qquad \tilde{s}(z) = ((z-p)/z) \in \tilde{\ell}^{1}(\rho).$$

As in the continuous-time case [1], write

$$\{\tilde{Q}(1/z)[I+\tilde{G}(z)]\tilde{P}(1/z)\}^{-1} = \tilde{M}(z)\tilde{N}(z)^{-1} .$$

Then using the facts above, lemma I and (22)-(24), a detailed calculation shows that, as a consequence of (C), $\tilde{N}(z)^{-1} \in \tilde{\ell}^{1}_{n\times n}(\rho)$; since $\tilde{M}(z)$ is also in this algebra, the claim follows. ¤

Necessity. By assumption $\tilde{H} \in \tilde{\ell}^{1}_{n\times n}(\rho)$.

(22) follows immediately by [6].

To establish (C) we use contradiction. We show that if the L.H.S. of (C) is zero, then there exists an input $u \in \ell^{2}_{n}(\rho)$ which results in an error e and thus an output $y = u-e$ which is not in $\ell^{2}_{n}(\rho)$. This is a contradiction because $u \in \ell^{2}_{n}(\rho)$ and $H \in \ell^{1}_{n\times n}(\rho)$ imply $y = H*u \in \ell^{2}_{n}(\rho)$ (This is an easy extension of lemma 1 of [2] where $\rho = 1$ has been handled). The z-transforms of e and u are related by

(25) $$[I+\tilde{G}(z)]\tilde{e}(z) = \tilde{u}(z).$$

Multiplying (25) on the left by $\tilde{Q}(1/z)$ and setting

(26) $$\tilde{N}(z) = \tilde{Q}(1/z)[I+\tilde{G}(z)]\tilde{P}(1/z)\tilde{M}(z)$$

(27) $$\overline{u}(z) = \tilde{Q}(1/z)\tilde{u}(z)$$

(28) $$\tilde{e}(z) = \tilde{P}(1/z)\tilde{M}(z)\overline{e}(z)$$

we obtain

(29) $$\tilde{N}(z)\overline{e}(z) = \overline{u}(z)$$

Observe that

(30) $$\tilde{M}(z) = \tilde{s}(z)^{m}\,\tilde{\Delta}(z)$$

where

(31) $$\tilde{\Delta}(z) = \mathrm{diag}\{\underbrace{1,1,\ldots,1}_{r_0},\ \underbrace{\tilde{s}(z)^{-c_{r_0+1}},\tilde{s}(z)^{-c_{r_0+1}},\ldots,\tilde{s}(z)^{-c_\alpha}}_{\alpha-r_0},\ \underbrace{\tilde{s}(z)^{-m},\tilde{s}(z)^{-m},\ldots,\tilde{s}(z)^{-m}}_{n-\alpha}\}.$$

With

(32) $$\tilde{N}_1(z) = \tilde{Q}(1/z)\left(\sum_{i=0}^{m-1} R_i(z-p)^{-m+i}\right)\tilde{P}(1/z)\tilde{M}(z)$$

(33) $$\tilde{N}_2(z) = \tilde{Q}(1/z)[I+\tilde{G}_0(z)]\tilde{P}(1/z)\tilde{M}(z)$$

(26) and (11) imply

$$\tilde{N}(z) = \tilde{N}_1(z) + \tilde{N}_2(z). \tag{34}$$

By (32), (30), (31), (24), (12), (13), (15) and (18)

$$\tilde{N}_1(z) = \tilde{D}_1(z) \oplus \tilde{D}_2(z) \tag{35}$$

where

$$\tilde{D}_1(z) = \mathrm{diag}\{\underbrace{\tilde{a}_1(1/z), \tilde{a}_2(1/z), \ldots, \tilde{a}_{r_0}(1/z)}_{r_0}, \underbrace{\tilde{b}_{r_0+1}(1/z), \tilde{b}_{r_0+2}(1/z), \ldots, \tilde{b}_{\alpha}(1/z)}_{\alpha-r_0}\} \tag{36}$$

$$\tilde{D}_2(z) = \tag{37}$$

$$\mathrm{diag}\{\underbrace{\tilde{b}_{\alpha+1}(1/z), \tilde{b}_{\alpha+2}(1/z), \ldots, \tilde{b}_{\beta}(1/z)}_{\beta-\alpha}, \underbrace{\tilde{b}_{\beta+1}(1/z)\tilde{s}(z)^{c_{\beta+1}-m}, \tilde{b}_{\beta+2}(1/z)\tilde{s}(z)^{c_{\beta+2}-m}, \ldots, b_r(1/z)s(z)^{c_r-m}}_{r-\beta}, \underbrace{0, 0, \ldots, 0}_{n-r}\}$$

By (33), (20), (23) and (24)

$$\tilde{N}_2(z) = \begin{array}{c c} & \begin{array}{cc} \overbrace{\hphantom{\tilde{K}_{11}(z)}}^{\alpha} & \overbrace{\hphantom{\tilde{L}_{12}(z)}}^{n-\alpha} \end{array} \\ \begin{array}{c} \alpha\,\{ \\ n-\alpha\,\{ \end{array} & \left[\begin{array}{c|c} \tilde{K}_{11}(z) & \tilde{L}_{12}(z) \\ \hline \tilde{K}_{21}(z) & \tilde{L}_{22}(z) \end{array}\right] \end{array} \tag{38}$$

where

(39) $$\tilde{K}_{11}(p) = 0$$

(40) $$\tilde{K}_{21}(p) = 0 \quad .$$

Furthermore by (36), (14) and (16)

(41) $$\det \tilde{D}_1(p) \neq 0$$

and by (37), (16), (24) and (19)

(42) "(C) not true" is equivalent to $\det(\tilde{D}_2(p) + \tilde{L}_{22}(p)) = 0$.

In order to establish the contradiction, using (42) we can pick a <u>nonzero</u> vector $\eta \in \mathbb{C}^{n-\alpha}$ in the null space of $\tilde{D}_2(p) + \tilde{L}_{22}(p)$, thus

(43) $$[\tilde{D}_2(p) + \tilde{L}_{22}(p)]\eta = 0.$$

Pick now the vector $\xi \in \mathbb{C}^{\alpha}$ such that

(44) $$\xi = - [\tilde{D}_1(p)]^{-1} \tilde{L}_{12}(p)\eta$$

which is well-defined because of (41) and because all elements of $\tilde{L}_{12}(z)$ are members of $\tilde{\ell}^1(\rho)$. Hence, setting

(45) $$\overline{e}(z) = \frac{z}{z-p}\begin{bmatrix}\xi \\ \eta\end{bmatrix}$$

$$\overline{u}(z) = \begin{bmatrix}\overline{u}_1(z) \\ \overline{u}_2(z)\end{bmatrix}\begin{matrix}\}\alpha \\ \}n-\alpha\end{matrix}$$

(29), (34), (35) and (38) imply

$$(46) \qquad \bar{u}_1(z) = \{[\tilde{D}_1(z) + \tilde{K}_{11}(z)]\xi + \tilde{L}_{12}(z)\eta\}(z/(z-p))$$

$$(47) \qquad \bar{u}_2(z) = \{\tilde{K}_{21}(z)\xi + [\tilde{D}_2(z) + \tilde{L}_{22}(z)]\eta\}(z)(z-p)).$$

Observe that because of (39)-(40) and (43)-(44) the expressions between the braces in the R.H.S. of (46)-(47) have a first order zero at p. Hence, since all elements of the matrices contained in these expressions belong to $\tilde{\ell}^1(\rho)$, $\bar{u}(z)$ is well-behaved and bounded at p. These remarks and the properties of the components of $\bar{u}_1(z)$ and $\bar{u}_2(z)$ imply that $\bar{u}(z)$ is analytic for $|z| > \rho$, bounded for $|z| \geq \rho$, continuous for $|z| = \rho$ and as $|z| \to \infty$, $\bar{u}(z) \to \bar{u}_0$ a finite constant vector. The Laurent expansion of $\bar{u}(z)$ about $z = 0$, [8, Sec. 9.14], reads

$$u(z) = \sum_{k=0}^{\infty} \bar{u}_k z^{-k} \quad \text{for } |z| > \rho;$$

using the uniform continuity of $\bar{u}$ in the compact annulus $\rho \leq |z| \leq \rho + 1$ we obtain

$$\bar{u}_k = \frac{\rho^k}{2\pi} \int_{-\pi}^{\pi} \bar{u}(\rho e^{j\theta}) e^{jk\theta} d\theta.$$

Observe that the $\bar{u}_k \rho^{-k}$ $k = 0,1,2,\ldots$ are Fourrier coefficients of $\bar{u}(\rho e^{j\theta})$ on $[-\pi,\pi]$. Now, since $\frac{\bar{u}(\rho e^{j\theta})}{\sqrt{2\pi}}$ is bounded and continuous on $[-\pi,\pi]$, it follows that $\frac{\bar{u}(\rho e^{j\theta})}{\sqrt{2\pi}}$ belongs to the Hilbertspace $L_n^2[-\pi,\pi]$. Furthermore the set $\{\frac{e^{jk\theta}}{\sqrt{2\pi}}\}_{k=-\infty}^{\infty}$ is an orthonormal Hilbert basis for $L^2[-\pi,\pi]$. Hence by Parseval's equality, [8,Sec. 6.5], $\sum_{k=0}^{\infty} |\bar{u}_k|^2 \rho^{-2k} = \frac{1}{2\pi}\int_{-\pi}^{\pi} |\bar{u}(\rho . e^{j\theta})|^2 d\theta < \infty$.

It follows therefore that $\bar{u}(z) \in \tilde{\ell}^2(\rho)$. Furthermore by (27) and fact 3 $\tilde{u}(z) \in \tilde{\ell}^2_n(\rho)$ or

(48) $\qquad u \in \ell^2_n(\rho).$

Finally by (45), (28), (23)-(24) and since $\tilde{P}(1/z)$ is unimodular, $\tilde{e}(z)/z$ has a pole at p with nonzero residue. Since $|p| \geq \rho$

(49) $\qquad e \notin \ell^2_n(\rho)$

and by (48)-(49) we have shown the contradiction we were after. ¤

Remarks.

1) The theorem above describes in detail what happens when K_-, K_0, K_+ are nonempty. When one or more of these sets are empty the required modifications of (C) of the multiplier $\tilde{M}(z)$ are straightforward.

2) In case there are ℓ poles at $p_1, p_2, \ldots, p_\ell$ of order $m_1, m_2, \ldots, m_\ell$ with absolute value larger than or equal to ρ, one uses a product of multipliers like $\tilde{M}(z)$ one for each pole. Condition (C) is used to check that det $\tilde{N}(z)$ does not vanish at $z = \rho$. Therefore for the more general case an appropriate condition (C) is required at each pole.

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